Construction of Hyperfunction Solutions to Invariant Linear Differential Equations (Asymptotic Analysis and Microlocal Analysis of PDE)

Muro, Masakazu

数理解析研究所講究録 1211 巻 2001 年 143-154

「微分方程式の漸近解析・超局所解析」研究集会

Construction of Hyperfunction Solutions to Invariant Linear Differential Equations.

(October 19, 2000)

Masakazu Muro (Gifu University)

Abstract

Constructions of invariant hyperfunction solutions of invariant linear differential equations with polynomial coefficients on some vector spaces V with actions of Lie groups G are discussed in this talk. We shall deal with the vector space of $n \times n$ real symmetric matrices, and those of complex and quaternion Hermitian matirices, on which the real, the complex and the quaternion general linear groups of degree n naturally act on these vector spaces, respectively. For a subgroup G in the general linear group, we observe in the main theorem that every invariant hyperfunction solution is expressed as a linear combination of Laurent expansion coefficients of a complex power of the determinant function with respect to the power parameter. Then the problem can be reduced to the determination of Laurent expansion coefficients which are needed to express the solution. We can give an algorithm to determine them. By applying the algorithm, we can prove that every invariant hyperfunction solutions to $\det(x)u(x) = 0$ is written as a sum of invarinat measures on the G-orbits in the set $S := \{x \in V \mid \det(x) = 0\}$ as one example. Some other examples are also given.

1 Introduction.

Let V be a real vector space on which a real algebraic subgroup G in $\mathbf{GL}(V)$ acts. Let $D(V)$ be the algebra of linear differential operators on V with polynomial coefficients and let $\mathcal{B}(V)$ be the space of hyperfunctions on V. We denote by $D(V)^G$ and $\mathcal{B}(V)^G$ the subspaces of G-invariant linear differential operators and of G-invariant hyperfunctions on V, respectively. For a given invariant differential operator $P(x, \partial) \in D(V)^G$ and an invariant hyperfunction $v(x) \in \mathcal{B}(V)^G$, we consider the linear differential equation

$$P(x, \partial)u(x) = v(x) \tag{1}$$

where the unknown function $u(x)$ is in $\mathcal{B}(V)^G$. In particular, our problem of this paper is the following: let $P(x, \partial) \in D(V)^G$ be a given *G-invariant* and *homogeneous* (see Definition 2.1) differential operator. Construct a basis of G-invariant hyperfunction solutions $u(x) \in \mathcal{B}(V)^G$ to the differential equation

$$P(x, \partial)u(x) = 0.$$

In this talk, we consider the problem in the following three cases. We prove Theorem 4.1 and determine the G-invariant kernel of $P(x, \partial)$ in some typical cases. Similar problems were considered by P.-D. Methée [5], [6] and [7] for Lorentz group invariant differential equations.

1. **real symmetric matrix space:** Let $V := \mathrm{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\mathbf{GL}_n(\mathbb{R})$ be the general linear group over $\mathbb{R}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{R})$ acts on the vector space V by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g, \tag{2}$$

with $x \in V$ and $g \in \mathbf{GL}_n(\mathbb{R})$. Then the subgroup

$$G := \{g \in \mathbf{GL}_n(\mathbb{R}) \mid \det(g \cdot {}^t g) = 1\} \tag{3}$$

acts on V naturally. Here ${}^t g$ means the transposed matrix of g.

2. **complex Hermitian matrix space:** Let $V := \mathrm{Her}_n(\mathbb{C})$ be the space of $n \times n$ Hermitian matrices over the complex field $\mathbb{C}$ and let $\mathbf{GL}_n(\mathbb{C})$ be the special linear group over $\mathbb{R}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{C})$ acts on the vector space V by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t\overline{g}, \tag{4}$$

with $x \in V$ and $g \in \mathbf{GL}_n(\mathbb{C})$. Then the subgroup

$$G := \{g \in \mathbf{GL}_n(\mathbb{C}) \mid \det(g \cdot {}^t\overline{g}) = 1\} \tag{5}$$

acts on V naturally. Here ${}^t\overline{g}$ means the transposed matrix of the complex conjugate of g. The determinant function $P(x) := \det(x)$ on $x \in \mathrm{Her}_n(\mathbb{C})$ is a real-valued irreducible polynomial.

3. **quaternion Hermitian matrix space:** Let $V := \mathrm{Her}_n(\mathbb{H})$ be the space of $n \times n$ Hermitian matrices over the quaternion field $\mathbb{H}$ and let $\mathbf{GL}_n(\mathbb{H})$ be the general linear group over $\mathbb{H}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{H})$ acts on the vector space V by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t\overline{g}, \tag{6}$$

with $x \in V$ and $g \in \mathbf{GL}_n(\mathbb{H})$. Then the subgroup

$$G := \{g \in \mathbf{GL}_n(\mathbb{H}) \mid \det(g \cdot {}^t\overline{g}) = 1\} \tag{7}$$

acts on V naturally. Here ${}^t\overline{g}$ means the transposed matrix of the quaternion conjugate of g. The determinant function $P(x) := \det(x)$ on $x \in \mathrm{Her}_n(\mathbb{H})$ is defined as a Pffafian of a $2n \times 2n$ complex alternating matrix and it is a real-valued irreducible polynomial.

2 Algebra of Invariant Differential Operators.

First we consider the case of $V := \mathrm{Sym}_n(\mathbb{R})$. Let $x \in \mathrm{Sym}_n(\mathbb{R})$. By using the upper half entries of x, we denote by

$$x = (x_{ij})_{n \geq j \geq i \geq 1}, \quad \partial = (\partial_{ij})_{n \geq j \geq i \geq 1} = \left(\frac{\partial}{\partial x_{ij}}\right)_{n \geq j \geq i \geq 1}$$

the coordinte and the prartial differentials on $\mathrm{Sym}_n(\mathbb{R})$, and by

$$x^\alpha = \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta = \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}$$

their integer powers where

$$\alpha = (\alpha_{ij}) \in \mathbb{Z}^m_{\geq 0}, \quad |\alpha| = \sum_{n \geq j \geq i \geq 1} \alpha_{ij}$$
$$\beta = (\beta_{ij}) \in \mathbb{Z}^m_{\geq 0}, \quad |\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}$$

and $m = n(n+1)/2$. The symbols x and ∂ also express

$$x = (x_{ij})_{n \geq j, i \geq 1} \in \mathrm{Sym}_n(\mathbb{C}[V]) \subset \mathrm{Sym}_n(D(V)),$$
$$\partial = (\partial_{ij})_{n \geq j, i \geq 1} \in \mathrm{Sym}_n(D(V)),$$

respectively, by consdering $x_{ij} = x_{ji}$. On the other hand, we also define $\partial^* \in \mathrm{Sym}_n(D(V))$ by

$$\partial^* = (\partial^*_{ij})_{n \geq i, j \geq 1} \quad \text{where} \quad \partial^*_{ij} := \begin{cases} \partial_{ij} & i = j, \\ \frac{1}{2}\partial_{ij} & i \neq j. \end{cases} \tag{8}$$

Next we consider the cases of $V := \mathrm{Her}_n(\mathbb{C})$ and $V := \mathrm{Her}_n(\mathbb{H})$. Let $x = (x_{ij})_{1\leq i,j\leq n} \in \mathrm{Her}_n(\mathbb{C})$ where

$$x_{ij} = x_{ij}^{(0)} + \sqrt{-1}x_{ij}^{(1)} \in \mathbb{C} \quad \text{and} \quad \overline{x_{ij}} = x_{ji} \tag{9}$$

with $x_{ij}^{(0)}, x_{ij}^{(1)} \in \mathbb{R}$ for $1 \leq i \leq j \leq n$ in the complex case and let $x = (x_{ij})_{1\leq i,j\leq n} \in \mathrm{Her}_n(\mathbb{H})$ where

$$x_{ij} = x_{ij}^{(0)} + x_{ij}^{(1)}\boldsymbol{i} + x_{ij}^{(2)}\boldsymbol{j} + x_{ij}^{(3)}\boldsymbol{k} \in \mathbb{H} \quad \text{and} \quad \overline{x_{ij}} = x_{ji} \tag{10}$$

with $x_{ij}^{(0)}, x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)} \in \mathbb{R}$ for $1 \leq i \leq j \leq n$ in the quaternion case. Here $\sqrt{-1}$ is the imaginary unit in $\mathbb{C}$ and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the imaginary units in $\mathbb{H}$, i.e., $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. $\overline{x_{ij}}$ means the complex and quaternion conjugate of x_{ij}, respectively.

In the complex case, by using the upper half triangular entries of x, we denote by

$$x = ((x_{ij}^{(0)})_{n\geq j\geq i\geq 1}, (x_{ij}^{(1)})_{n\geq j>i\geq 1}),$$
$$\partial = ((\partial_{ij}^{(0)})_{n\geq j\geq i\geq 1}, (\partial_{ij}^{(1)})_{n\geq j>i\geq 1})$$

with $\partial_{ij}^{(k)} = \left(\dfrac{\partial}{\partial x_{ij}^{(k)}}\right)$, the coordinte and the partial differentials on $\mathrm{Her}_n(\mathbb{C})$, and by

$$x^\alpha = \prod_{n\geq j\geq i\geq 1} (x_{ij}^0)^{\alpha_{ij}^{(0)}} \times \prod_{n\geq j>i\geq 1} (x_{ij}^1)^{\alpha_{ij}^{(1)}},$$
$$\partial^\beta = \prod_{n\geq j\geq i\geq 1} (\partial_{ij}^{(0)})^{\beta_{ij}^{(0)}} \times \prod_{n\geq j>i\geq 1} (\partial_{ij}^{(1)})^{\beta_{ij}^{(1)}}$$

their integer powers where

$$\alpha = (\alpha_{ij}^{(0)}, \alpha_{ij}^{(1)}) \in \mathbb{Z}_{\geq 0}^m, \quad |\alpha| = \sum_{n\geq j\geq i\geq 1} \alpha_{ij}^{(0)} + \sum_{n\geq j>i\geq 1} \alpha_{ij}^{(1)}$$
$$\beta = (\beta_{ij}^{(0)}, \beta_{ij}^{(1)}) \in \mathbb{Z}_{\geq 0}^m, \quad |\beta| = \sum_{n\geq j\geq i\geq 1} \beta_{ij}^{(0)} + \sum_{n\geq j>i\geq 1} \beta_{ij}^{(1)}$$

with $m = (n(n+1)/2) + (n(n-1)/2) = n^2$. The symbols x and ∂ also express the Hermitian matrices on $D(V)$

$$x = (x_{ij}^{(0)})_{n\geq j,i\geq 1} + \sqrt{-1}(x_{ij}^{(1)})_{n\geq j,i\geq 1} \in \mathrm{Her}_n(\mathbb{C}[V]) \subset \mathrm{Her}_n(D(V)),$$
$$\partial = (\partial_{ij}^{(0)})_{n\geq j,i\geq 1} + \sqrt{-1}(\partial_{ij}^{(1)})_{n\geq j,i\geq 1} \in \mathrm{Her}_n(D(V)),$$

respectively, by considering $x_{ij}^{(0)} = x_{ji}^{(0)}$ and $x_{ij}^{(1)} = -x_{ji}^{(1)}$. On the other hand, we also define $\partial^* \in \mathrm{Her}_n(D(V))$ by

$$\partial^* = (\partial_{ij}^{(0)*})_{n\geq i,j\geq 1} + \sqrt{-1}(\partial_{ij}^{(1)*})_{n\geq i,j\geq 1} \ \text{ where } \ \partial_{ij}^{(k)*} := \begin{cases} \partial_{ij}^{(0)} & i=j, k=0, \\ 0 & i=j, k=1, \\ \frac{1}{2}\partial_{ij}^{(k)} & i\neq j, k=0,1. \end{cases} \tag{11}$$

In the quaternion case, by using the upper half triangular entries of x, we denote by

$$x = ((x_{ij}^{(0)})_{n\geq j\geq i\geq 1}, (x_{ij}^{(1)})_{n\geq j>i\geq 1}, (x_{ij}^{(2)})_{n\geq j>i\geq 1}, (x_{ij}^{(3)})_{n\geq j>i\geq 1},),$$
$$\partial = ((\partial_{ij}^{(0)})_{n\geq j\geq i\geq 1}, (\partial_{ij}^{(1)})_{n\geq j>i\geq 1}, (\partial_{ij}^{(2)})_{n\geq j>i\geq 1}, (\partial_{ij}^{(3)})_{n\geq j>i\geq 1},)$$

with $\partial_{ij}^{(k)} = \left(\dfrac{\partial}{\partial x_{ij}^{(k)}}\right)$, the coordinte and the partial differentials on $\mathrm{Her}_n(\mathbb{H})$, and by

$$x^\alpha = \prod_{n\geq j\geq i\geq 1} (x_{ij}^0)^{\alpha_{ij}^{(0)}} \times \prod_{\substack{n\geq j>i\geq 1\\ k=1,2,3}} (x_{ij}^k)^{\alpha_{ij}^{(k)}},$$
$$\partial^\beta = \prod_{n\geq j\geq i\geq 1} (\partial_{ij}^{(0)})^{\beta_{ij}^{(0)}} \times \prod_{\substack{n\geq j>i\geq 1\\ k=1,2,3}} (\partial_{ij}^{(k)})^{\beta_{ij}^{(k)}}$$

their integer powers where

$$\alpha = (\alpha_{ij}^{(0)}, \alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \alpha_{ij}^{(3)}) \in \mathbb{Z}_{\geq 0}^{m}, \quad |\alpha| = \sum_{n \geq j \geq i \geq 1} \alpha_{ij}^{(0)} + \sum_{\substack{n \geq j > i \geq 1 \\ k=1,2,3}} \alpha_{ij}^{(k)}$$

$$\beta = (\beta_{ij}^{(0)}, \beta_{ij}^{(1)}, \beta_{ij}^{(2)}, \beta_{ij}^{(3)}) \in \mathbb{Z}_{\geq 0}^{m}, \quad |\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}^{(0)} + \sum_{\substack{n \geq j > i \geq 1 \\ k=1,2,3}} \beta_{ij}^{(k)}$$

with $m = (n(n+1)/2) + 3(n(n-1)/2) = 2n^2 - n$. The symbols x and ∂ also express the Hermitian matrices on $D(V) \otimes \mathbb{H}$

$$\begin{aligned} x &= (x_{ij}^{(0)})_{n \geq j, i \geq 1} + i(x_{ij}^{(1)})_{n \geq j, i \geq 1} + j(x_{ij}^{(2)})_{n \geq j, i \geq 1} + k(x_{ij}^{(3)})_{n \geq j, i \geq 1} \\ &\in \mathrm{Her}_n(\mathbb{H}[V]) \subset \mathrm{Her}_n(D(V) \otimes \mathbb{H}), \\ \partial &= (\partial_{ij}^{(0)})_{n \geq j, i \geq 1} + i(\partial_{ij}^{(1)})_{n \geq j, i \geq 1} + j(\partial_{ij}^{(2)})_{n \geq j, i \geq 1} + k(\partial_{ij}^{(3)})_{n \geq j, i \geq 1} \\ &\in \mathrm{Her}_n(D(V) \otimes \mathbb{H}), \end{aligned}$$

respectively, by consdering $x_{ij}^{(0)} = x_{ji}^{(0)}$ and $x_{ij}^{(k)} = -x_{ji}^{(k)}$ for $k = 1, 2, 3$. On the other hand, we also define $\partial^* \in \mathrm{Her}_n(D(V) \otimes \mathbb{H})$ by

$$\partial^* = (\partial_{ij}^{(0)*})_{n \geq i, j \geq 1} + i(\partial_{ij}^{(1)*})_{n \geq i, j \geq 1} + j(\partial_{ij}^{(2)*})_{n \geq i, j \geq 1} + k(\partial_{ij}^{(3)*})_{n \geq i, j \geq 1} \tag{12}$$

where

$$\partial_{ij}^{(k)*} := \begin{cases} \partial_{ij}^{(0)} & i = j, k = 0, \\ 0 & i = j, k = 1, 2, 3, \\ \frac{1}{2}\partial_{ij}^{(k)} & i \neq j, k = 0, 1, 2, 3. \end{cases}$$

Definition 2.1 (order and homogeneous degree). For vector spaces $V = \mathrm{Sym}_n(\mathbb{R}), \mathrm{Her}_n(\mathbb{C})$ and Her any differential operator $P(x, \partial) \in D(V)$ is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^{m} \\ |\beta| = k}} a_{\alpha\beta} x^{\alpha} \partial^{\beta}. \tag{13}$$

We call the *order of* $P(x, \partial)$ the highest number k in the sum (13). On the other hand, for

$$P(x, \partial) := \sum_{k \in \mathbb{Z}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^{m} \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^{\alpha} \partial^{\beta} \tag{14}$$

The differential operator $\sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^{m} \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^{\alpha} \partial^{\beta}$ in (14) is called the *homogeneous part* of $P(x, \partial)$ of degree k. A differential operator with only one homogeneous part of degree k is called *a homogeneous differential operator* and we say that k is the *homogeneous degree.*

Example 2.1 (generators of invariant differential operators). Let $V = \mathrm{Sym}_n(\mathbb{R})$ (resp. $V = \mathrm{Her}_n(\mathbb{C})$, $V = \mathrm{Her}_n(\mathbb{H})$). Then we can construct G-invariant differential operators $\{P_k(x, \partial)\}_{k=1,\dots,n}$ on V, which form a complete set of generaotrs of $\mathbf{GL}_n(\mathbb{R})$-invariant (resp. $\mathbf{GL}_n(\mathbb{C})$-invariant, $\mathbf{GL}_n(\mathbb{H})$-invariant) differential operators.

1. Let h and n be positive integers with $1 \leq h \leq n$. A sequence of increasing integers $p = (p_1, \dots, p_h) \in \mathbb{Z}^h$ is called an *increasing sequence in* $[1, n]$ *of length* h if it satisfies $1 \leq p_1 < \cdots < p_h \leq n$. We denote by $IncSeq(h, n)$ the set of increasing sequences in $[1, n]$ of length h.

2. For two sequences $p = (p_1, \dots, p_h)$ and $q = (q_1, \dots, q_h) \in IncSeq(h,n)$ and for an $n \times n$ symmetric matrix $x = (x_{ij}) \in \mathrm{Sym}_n(\mathbb{R})$, (resp. complex Hermitian matrix $x = (x_{ij}^{(0)} + \sqrt{-1}x_{ij}^{(1)}) \in \mathrm{Her}_n(\mathbb{C})$, quaternion Hermitian matrix $x = (x_{ij}^{(0)} + ix_{ij}^{(1)} + jx_{ij}^{(2)} + kx_{ij}^{(3)}) \in \mathrm{Her}_n(\mathbb{H})$), we define an $h \times h$ matrix $x_{(p,q)}$ by

$$\begin{aligned} x_{(p,q)} &:= (x_{p_i,q_j})_{1 \le i \le j \le h} \\ (resp.\ x_{(p,q)} &:= (x^{(0)}_{p_i,q_j} + \sqrt{-1}x^{(1)}_{p_i,q_j})_{1 \le i \le j \le h}, \\ x_{(p,q)} &:= (x^{(0)}_{p_i,q_j} + ix^{(1)}_{p_i,q_j} + jx^{(2)}_{p_i,q_j} + kx^{(3)}_{p_i,q_j})_{1 \le i \le j \le h}). \end{aligned}$$

In the same way, for an $n \times n$ real symmetric (resp. complex Hermitian, quaternion Hermitian) matrix $\partial = (\partial_{ij})$ (resp. $\partial^* = (\partial_{ij}^{(0)*} + \sqrt{-1}\partial_{ij}^{(1)*})$, $\partial^* = (\partial_{ij}^{(0)*} + i\partial_{ij}^{(1)*} + j\partial_{ij}^{(2)*} + k\partial_{ij}^{(3)*})$) of differential operators, we define an $h \times h$ matrix $\partial^*_{(p,q)}$ of differential operators by

$$\begin{aligned} \partial^*_{(p,q)} &:= (\partial^*_{p_i,q_j})_{1 \le i \le j \le h} \\ (resp.\ \partial^*_{(p,q)} &:= (\partial^{(0)*}_{p_i,q_j} + \sqrt{-1}\partial^{(1)*}_{p_i,q_j})_{1 \le i \le j \le h}, \\ \partial^*_{(p,q)} &:= (\partial^{(0)*}_{p_i,q_j} + i\partial^{(1)*}_{p_i,q_j} + j\partial^{(2)*}_{p_i,q_j} + k\partial^{(3)*}_{p_i,q_j})_{1 \le i \le j \le h}). \end{aligned}$$

3. For an integer h with $1 \le h \le n$, we define

$$P_h(x, \partial) := \sum_{p,q \in IncSeq(h,n)} \det(x_{(p,q)}) \det(\partial^*_{(p,q)}). \tag{15}$$

4. In particular, $P_n(x, \partial) = \det(x)\det(\partial^*)$ and *Euler's differential operator* is given by

$$P_1(x, \partial) = \sum_{n \ge j \ge i \ge 1} x_{ij} \frac{\partial}{\partial x_{ij}} = \mathrm{tr}(x \cdot \partial^*). \tag{16}$$

These are all homogeneous differential operators of degree 0 and invariant under the action of $\mathbf{GL}(V)$, and hence it is also invariant under the action of $G \subset \mathbf{GL}(V)$.

5. $\det(x)$ and $\det(\partial^*)$ are homogeneous differential operators of degree n and $-n$, respectively. They are invariant under the action of G, and relatively invariant differential operators under the action of $\mathbf{GL}_n(\mathbb{R})$ (resp. $\mathbf{GL}_n(\mathbb{C})$, $\mathbf{GL}_n(\mathbb{H})$), with characters $\chi(g) := \det(g \cdot {}^t\overline{g})$ and $\chi^{-1}(g) := \det(g \cdot {}^t\overline{g})^{-1}$, respectively.

Proposition 2.1. *Let* $V = \mathrm{Sym}_n(\mathbb{R})$ *(resp.* $V = \mathrm{Her}_n(\mathbb{C})$, $V = \mathrm{Her}_n(\mathbb{H})$*).*

1. Every $\mathbf{GL}_n(\mathbb{R})$*-invariant (resp.* $\mathbf{GL}_n(\mathbb{C})$*-invariant,* $\mathbf{GL}_n(\mathbb{H})$*-invariant) differential operator on* V *can be expressed as a polynomial in* $P_i(x, \partial) (i = 1, \dots, n)$ *defined in* (15).

2. Every G*-invariant differential operator on* V *can be expressed as a polynomial in* $P_i(x, \partial)$ $(i = 1, \dots, n-1)$, $\det(x)$ *and* $\det(\partial^*)$.

Proof. We can give the proof almost in the same way as the proof of H. Maass [4] in the case of symmetric matrices. See also Nomura [13] and [14]. □

3 Complex Powers of the Determinant Functions.

We consider the case of $V := \mathrm{Sym}_n(\mathbb{R})$ (resp. $V := \mathrm{Her}_n(\mathbb{C})$, $V := \mathrm{Her}_n(\mathbb{H})$). We denote $P(x) := \det(x)$ and we set $S := \{x \in V | \det(x) = 0\}$. The subset $V - S$ decomposes into $n+1$ connected components,

$$\begin{aligned} V_i &:= \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (i, n-i)\} \\ (resp.\ V_i &:= \{x \in \mathrm{Her}_n(\mathbb{C}) \mid \mathrm{sgn}(x) = (2i, 2(n-i))\} \\ V_i &:= \{x \in \mathrm{Her}_n(\mathbb{H}) \mid \mathrm{sgn}(x) = (4i, 4(n-i))\}) \end{aligned} \tag{17}$$

with $i = 0, 1, \ldots, n$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Sym}_n(\mathbb{R})$ (resp. $\mathrm{sgn}(x)$ for $x \in \mathrm{Her}_n(\mathbb{C})$, $\mathrm{sgn}(x)$ for $x \in \mathrm{Her}_n(\mathbb{H})$) is the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \overline{\vec{v}}$ on $\vec{v} \in \mathbb{R}^n$ (resp. $\vec{v} \in \mathbb{C}^n$, $\vec{v} \in \mathbb{H}^n$). We define the complex power function of $P(x)$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in V_i, \\ 0 & \text{if } x \notin V_i. \end{cases} \tag{18}$$

for a complex number $s \in \mathbb{C}$. We consider a linear combination of the hyperfunctions $|P(x)|_i^s$

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s \tag{19}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

Definition 3.1 (Laurent expansion coefficients). Let $\vec{a} \in \mathbb{C}^{n+1}$ and suppose that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = \lambda$. Then we have the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = \lambda$,

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},\lambda]}(x)\,(s-\lambda)^w . \tag{20}$$

We often denote by

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a},s]}(x)) := P_w^{[\vec{a},\lambda]}(x) \tag{21}$$

the w-th Laurent expansion coefficient of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ in (20).

Proposition 3.1. *Let $V := \mathrm{Sym}_n(\mathbb{R})$ (resp. $V := \mathrm{Her}_n(\mathbb{C})$, $V := \mathrm{Her}_n(\mathbb{H})$). Then $P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -(k+1)/2$ (resp. $s = -k$, $s = -k$) with $k = 1, 2, \ldots$. The possible highest order of the pole of $P^{[\vec{a},s]}(x)$ at $s = -(k+1)/2$ (resp. $s = -k$, $s = -k$) is*

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & (k = 1, 2. \ldots, n-1), \\ \lfloor \frac{n}{2} \rfloor & (k = n, n+1. \ldots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & (k = n, n+1. \ldots, \text{ and } k+n \text{ is even}). \end{cases}$$
$$\left(resp. \begin{cases} k & (k = 1, 2. \ldots, n-1), \\ n & (k = n, n+1. \ldots). \end{cases}, \quad \begin{cases} \lfloor \frac{k+1}{2} \rfloor & (k = 1, 2. \ldots, 2n-1), \\ n & (k = 2n, 2n+1. \ldots). \end{cases} \right) \tag{22}$$

Proposition 3.2. *Let $V := \mathrm{Sym}_n(\mathbb{R})$ (resp. $V := \mathrm{Her}_n(\mathbb{C})$, $V := \mathrm{Her}_n(\mathbb{H})$). Let $P(x,\partial) \in D(V)^G$ be a homogeneous differential operator.*

1. *The homogeneous degree of $P(x,\partial)$ is in $(n \cdot \mathbb{Z})$. Namely the homogeneous degree is divisible by n.*

2. *If the homogeneous degree of $P(x,\partial)$ is nk with $k \in \mathbb{Z}$, then we have*

$$P(x,\partial)(\det x)^s = b_P(s)(\det x)^{s+k} \tag{23}$$

where $b_P(s)$ is a polynomial in $s \in \mathbb{C}$ and $x \in \mathrm{Sym}_n(\mathbb{R})$ is positive definite. We have also

$$\begin{aligned} P(x,\partial)P^{[\vec{a},s]}(x) &= b_P(s)\det(x)^k P^{[\vec{a},s]}(x) \\ &= b_P(s)\mathrm{sgn}(\det(x))^k P^{[\vec{a},s+k]}(x) \end{aligned} \tag{24}$$

for all $x \in V - S$.

3. *If $k < 0$, then $b^{\underline{-k}}(s-1) | b_P(s)$ where $b^{\underline{-k}}(s-1) := b(s-1)b(s-2)\cdots b(s-(-k))$ with $b(s) := \prod_{i=1}^{n}(s + \frac{i+1}{2})$ (resp. $b(s) := \prod_{i=1}^{n}(s+i)$, $b(s) := \prod_{i=1}^{n}(s+2i-1)$).*

Definition 3.2 (b_P-function). Let $P(x,\partial) \in D(V)^G$ be a homogeneous differential operator. We call $b_P(s)$ in (23) the *b_P-function* of $P(x,\partial)$. Namely let $P(x,\partial)$ be a G-invariant homogeneous differential operator of homogeneous degree nk ($k \in \mathbb{Z}$). (Homogeneous degree of G-invariant differential operator is divisible by n.) Then we have

$$P(x,\partial)(\det(x))^s = {}^{\exists}b_P(s)(\det(x))^{s+k}$$

with $s \in \mathbb{C}$ and $x > 0$,i.e., positive definite. Here $b_P(s)$ is a polynomial in $\mathbb{C}$. We call $b_P(s)$ the *b_P-function* of $P(x,\partial)$.

The b_P-functions are closely related to the b-functions treated in Kashiwara [3] but they have different properties. Kashiwara proved that the roots of b-functions are negative rational numbers. But the roots of b_P-functions may take any complex numbers.

Example 3.1. Let $V := \mathrm{Sym}_n(\mathbb{R})$ (resp. $V := \mathrm{Her}_n(\mathbb{C})$, $V := \mathrm{Her}_n(\mathbb{H})$).

1. For the invariant differential operator of homogeneous degree $kn = 0$

$$P(x,\partial) := \sum_{p,q \in IncSeq(h,n)} \det(x_{(p,q)}) \det(\partial^*_{(p,q)})$$

   defined by (15), we have

$$\begin{aligned} b_P(s) &= \text{const.}(s)(s+\frac{1}{2})\cdots(s+\frac{h-1}{2}) \\ (\textit{resp. } b_P(s) &= \text{const.}(s)(s+1)\cdots(s+h-1), \\ b_P(s) &= \text{const.}(s)(s+2)\cdots(s+2(h-1))) \end{aligned}$$

2. For $P(x,\partial) = \det(\partial^*)$ (homogeneous degree $kn = -n$),

$$\begin{aligned} b_P(s) &= \text{const.}(s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2}) \\ (\textit{resp. } b_P(s) &= \text{const.}(s)(s+1)\cdots(s+n-1), \\ b_P(s) &= \text{const.}(s)(s+2)\cdots(s+2(n-1))). \end{aligned}$$

3. For $P(x,\partial) = \det(x)$ (homogeneous degree $kn = n$),

$$b_P(s) = 1.$$

4 Main Theorem.

The key theorem of this talk is the following.

Theorem 4.1. *Let $V := \mathrm{Sym}_n(\mathbb{R})$, $V := \mathrm{Her}_n(\mathbb{C})$ or $V := \mathrm{Her}_n(\mathbb{H})$ and let $P(x,\partial) \in D(V)^G$ be a non-zero homogeneous differential operator with homogeneous degree kn. We suppose that*

$$\textit{the degree of } b_P(s) \;=\; \textit{the order of } P(x,\partial). \tag{25}$$

The space of G-invariant hyperfunction solutions of the differential equation $P(x,\partial)u(x) = 0$ is finite dimensional. The solutions $u(x)$ are given as finite linear combinations of Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ at some finite number of points in $s \in \mathbb{C}$.

In the following sections, we shall determine the G-invariant hyperfunction kernel of $P(x,\partial) \in D(V)^G$ in some typical examples.

5 Oerder of poles of complex powers.

From now on, we shall consider only the case of $V = \mathrm{Sym}_n(\mathbb{R})$. The same arguments are possible for other cases, $V = \mathrm{Her}_n(\mathbb{C})$ and $V = \mathrm{Her}_n(\mathbb{H})$.

The exact order of pole of the hyperfunction

$$P^{[\vec{a},s]}(x) = \sum_{i=0}^{n} a_i |P(x)|_i^s$$

palys an important role. In order to determine the exact pole of $P^{[\vec{a},s]}(x)$ at $s = s_0$, the author [11] introduced the coefficient vectors

$$d^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1} \tag{26}$$

with $k = 0, 1, \dots, n$ in [11]. The precise statement is given in Definition 5.1. Here, $(\mathbb{C}^{n+1})^*$ means the dual vector space of $\mathbb{C}^{n+1}$. Each element of $d^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$ depending on $s_0 \in \mathbb{C}$,i.e.,a linear map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$,

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \mathbb{C}. \tag{27}$$

We denote

$$\langle d^{(k)}[s_0], \vec{a} \rangle = (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \dots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \mathbb{C}^{n-k+1}. \tag{28}$$

Definition 5.1 (Coefficient vectors $d^{(k)}[s_0]$). Let s_0 be a *half-integer*, i.e., a rational number given by $q/2$ with an integer q. We define the *coefficient vectors* $d^{(k)}[s_0]$ $(k = 0, 1, \dots, n)$ by induction in the following way.

1. First, we set

$$d^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \dots, d_n^{(0)}[s_0]) \tag{29}$$

such that $\langle d_i^{(0)}[s_0], \vec{a} \rangle := a_i$ for $i = 0, 1, \dots, n$.

2. Next, we define $d^{(1)}[s_0]$ and $d^{(2)}[s_0]$ by

$$d^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \dots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^n, \tag{30}$$

with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$, and

$$d^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \dots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1}, \tag{31}$$

with $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$. Here,

$$\epsilon[s_0] := \begin{cases} 1 & ,(\text{if } s_0 \text{ is a strict half-integer}), \\ (-1)^{s_0+1} & ,(\text{if } s_0 \text{ is an integer}). \end{cases} \tag{32}$$

A *strict half-integer* means a rational number given by $q/2$ with an odd integer q.

3. Lastly, by induction on k, we define the coefficient vectors $d^{(k)}[s_0]$ for $k = 0, 1, \dots, n$ by

$$d^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \dots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l}, \tag{33}$$

with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$, and

$$d^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \dots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1}, \tag{34}$$

with $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

By using $d^{(k)}[s_0]$ in Definition 5.1, the author obtained an algorithm to compute the exact order of poles of $P^{[\vec{a},s]}(x)$ in [11]. In this section, we shall characterize the space

$$A(\lambda, q) := \{\vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a},s]}(x) \text{ has a pole of order} \leq q \text{ at } s = \lambda\}. \tag{35}$$

in terms of the coefficient vectors $d^{(k)}[\lambda]$.

Definition 5.2. We define the vector subspaces $D_{half}^{(l)}$, $D_{even}^{(l)}$ and $D_{odd}^{(l)}$ in $\mathbb{C}^{n+1}$.

1.

$$D_{half}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle d^{(2l+2)}[\lambda], \vec{a}\rangle = 0 \text{ for any strict half-integer } \lambda\}.$$

Note that $d^{(2l+2)}[\lambda]$ does not depend on the choice of λ if it is a half-integer.

2.

$$D_{odd}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle d^{(2l+1)}[\lambda], \vec{a}\rangle = 0 \text{ for any odd integer } \lambda\}.$$

$$D_{even}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle d^{(2l+1)}[\lambda], \vec{a}\rangle = 0 \text{ for any even integer } \lambda\}.$$

Note that $d^{(2l+1)}[\lambda]$ does not depend on the choice of λ if it is an odd integer or an even integer, respectively.

Proposition 5.1. *$D_{half}^{(l)}$, $D_{even}^{(l)}$ and $D_{odd}^{(l)}$ in $\mathbb{C}^{n+1}$ have the following properties.*

1. *We define $\vec{a}^{\#} := ((-1)^n a_0, (-1)^{n-1} a_1, \dots, a_n) \in \mathbb{C}^{n+1}$ for $\vec{a} = (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$. Then we have*

$$\vec{a} \in D_{odd}^{(l)} \iff \vec{a}^{\#} \in D_{even}^{(l)}$$

and

$$\vec{a} \in D_{half}^{(l)} \iff \vec{a}^{\#} \in D_{half}^{(l)}.$$

2. *Let l be an integer $0 \leq l < PHO(\lambda)$. The vector subspace $A(\lambda, l)$ defined by (35) is characterized as*

$$\vec{a} \in A(\lambda, l) \iff \begin{cases} \vec{a} \in D_{half}^{(l)} & \text{if } \lambda \text{ is a strict half-integer,} \\ \vec{a} \in D_{odd}^{(l)} & \text{if } \lambda \text{ is an odd integer,} \\ \vec{a} \in D_{even}^{(l)} & \text{if } \lambda \text{ is an even integer.} \end{cases} \tag{36}$$

In addition, we have $A(\lambda, PHO(\lambda)) = \mathbb{C}^{n+1}$. Here, we denote by $PHO(\lambda)$ the possible highest order of $P^{[\vec{a},s]}(x)$ at $s = \lambda$. Namely,

$$PHO(\lambda) = \begin{cases} \lfloor \frac{k+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = 1, 2. \dots, n-1), \\ \lfloor \frac{n}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = n, n+1. \dots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = n, n+1. \dots, \text{ and } k+n \text{ is even}), \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Proof. 1. is a direct consequence of the definition. 2. can be proved by the main theorem of the author's paper [11]. □

6 Examples.

We give three examples in the case of $V = \mathrm{Sym}_n(\mathbb{R})$. Some homogeneous differential equations generated by $\det(x)$ and $\det(\partial^*)$ are daelt with here.

6.1 The equations $\det(\partial^*)\det(x)u(x) = 0$ and $\det(x)\det(\partial^*)u(x) = 0$

First we consider two examples of differential equation of homogeneous degree 0. Let us consider the case of $P(x,\partial) = \det(\partial^*)\det(x)$. and $P(x,\partial) = \det(x)\det(\partial^*)$. Then the homogeneous degree of $P(x,\partial)$ is 0 and $b_P(s) = (s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2})$ and $b_P(s) = (s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2})$, respectively.

Proposition 6.1. *Consider the differential equation* $\det(\partial^*)\det(x)u(x) = 0$.

1. *The* $\mathbf{SL}_n(\mathbb{R})$*-invariant hyperfunction solution space to the differential equation* $\det(\partial^*)\det(x)u(x) = 0$ *is generated by*

$$\bigcup_{k=1}^{n}\left\{ Laurent^{(j)}_{s=-\frac{k+1}{2}}(P^{[\vec{a},s]}(x)) \;\middle|\; j = 0,1,\dots,\lfloor\tfrac{k+1}{2}\rfloor \text{ and } \vec{a} \in A(-\tfrac{k+1}{2}, j)\right\} \tag{38}$$

Here, $A(-\frac{k+1}{2}, j)$ *is a vector subspace of* $\mathbb{C}^{n+1}$ *defined by* (35). *Similarly, the* $\mathbf{SL}_n(\mathbb{R})$*-invariant hyperfunction solution space to the differential equation* $\det(x)\det(\partial^*)u(x) = 0$ *is generated by*

$$\bigcup_{k=1}^{n}\left\{ Laurent^{(j)}_{s=-\frac{k-1}{2}}(P^{[\vec{a},s]}(x)) \;\middle|\; j = 0,1,\dots,\lfloor\tfrac{k-1}{2}\rfloor \text{ and } \vec{a} \in A(-\tfrac{k-1}{2}, j)\right\} \tag{39}$$

2. *In particular, for* $k = 1,2,\dots,n,n+1,n+2$,

$$\left\{ Laurent^{(j)}_{s=-\frac{k-1}{2}}(P^{[\vec{a},s]}(x)) \;\middle|\; j = 0,1,\dots,\lfloor\tfrac{k+1}{2}\rfloor \text{ and } \vec{a} \in A(-\tfrac{k-1}{2}, j)\right\} \tag{40}$$

forms an $n+1$*-dimensional vector space generated by all the relatively invariant tempered distributions under the action of* $g \in \mathbf{GL}_n(\mathbb{R})$ *corresponding to the character* $\det(g)^{-k+1}$.

6.2 The equations $\det(x)u(x) = 0$

Let us consider the case of $P(x,\partial) = \det(x)$. Then the total homogeneous degree of $P(x,\partial)$ is n and $b_P(s) = 1$. We can prove by our algorithm that the G-invariant solution space of the differential equation $\det(x)u(x) = 0$ is generated by the G-invariant measures on all the singular orbits (i.e., G-orbits contained in $\det(x) = 0$), and hence, it is $\frac{n(n+1)}{2}$-dimensional (= the number of singular orbits). Here the G-invariant measure on each singular orbit is a relatively invariant hyperfunction. Namely we have the following proposition.

Proposition 6.2. *Consider the differential equation* $\det(x)u(x) = 0$.

1. *The* $\mathbf{SL}_n(\mathbb{R})$*-invariant hyperfunction solution space to the differential equation* $\det(x)u(x) = 0$ *is generated by*

$$\bigcup_{k=1}^{n}\left\{ Laurent^{(\lfloor\frac{k+1}{2}\rfloor)}_{s=-\frac{k+1}{2}}(P^{[\vec{a},s]}(x)) \;\middle|\; \vec{a} \in \mathbb{C}^{n+1}\right\} \tag{41}$$

2. *In particular, for* $k = 1,2,\dots,n$,

$$\left\{ Laurent^{(\lfloor\frac{k+1}{2}\rfloor)}_{s=-\frac{k+1}{2}}(P^{[\vec{a},s]}(x)) \;\middle|\; \vec{a} \in \mathbb{C}^{n+1}\right\} \tag{42}$$

forms an $(n+1-k)$*-dimensional vector space generated by the tempered distributions*

$$f(x) \longmapsto \int f(x)d\nu_k^j \quad (f(x) \in \mathcal{S}(V))$$

$(j = 0,1,\dots,n+1-k)$ *where* $d\nu_k^j$ *is the* $\mathbf{SL}_n(\mathbb{R})$*-invariant measure on*

$$S_k^j := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (j, n-k-j)\}$$

6.3 The equations $\det(\partial^*)u(x) = 0$

Similar argument is possible for the case of $P(x, \partial) = \det(\partial)$. operators. In this case, the total homogeneous degree of $P(x, \partial)$ is $(-n)$ and we see that $b_P(s) = \prod_{i=1}^n (s + \frac{i-1}{2})$. The solution space of $\det(\partial)u(x) = 0$ is just the Fourier transform of that of $\det(x)u(x) = 0$, and hence it is $\frac{n(n+1)}{2}$-dimensional and generated by relatively invariant hyperfunctions. We can construct them from the complex power of $\det(x)$

Proposition 6.3. *Consider the differential equation* $\det(\partial^*)u(x) = 0$.

1. *The* $\mathbf{SL}_n(\mathbb{R})$*-invariant hyperfunction solution space to the differential equation* $\det(\partial^*)u(x) = 0$ *is generated by*

$$\bigcup_{k=1}^{n} \left\{ Laurent^{(j)}_{s=-\frac{n-k}{2}} (P^{[\vec{a}_j, s]}((x))) \;\middle|\; j = 0, 1, \ldots, \lfloor \tfrac{n-k}{2} \rfloor \text{ and } \vec{a}_j \in D^{(j)}_* \right\} \tag{43}$$

Here, $D^{(j)}_*$ *is a vector subspace of* $\mathbb{C}^{n+1}$ *defined by Definition 5.2. The* $*$ *in* $D^{(j)}_*$ *is substituted half, even or odd according as* λ *is a strictly half integer, an even integer or an odd integer, respectively.*

2. *In particular, for* $k = 1, 2, \ldots, n$,

$$\left\{ Laurent^{(j)}_{s=-\frac{n-k}{2}} (P^{[\vec{a}_j, s]}((x))) \;\middle|\; j = 0, 1, \ldots, \lfloor \tfrac{n-k}{2} \rfloor \text{ and } \vec{a}_j \in D^{(j)}_* \right\} \tag{44}$$

forms an $(n + 1 - k)$*-dimensional vector space generated by the Fourier transforms of the tempered distributions in* (42).

References

[1] L. Gårding, *The solution of Cauchy's problem for two totally hyperbolic differential equations by means of Riesz integrals*, Ann. of Math. **48** (1947), 785–826.

[2] I.M. Gelfand and G.E. Shilov, *Generalized Functions — properties and operations*, Generalized Functions, vol. 1, Academic Press, New York and London, 1964.

[3] M. Kashiwara, *B-functions and Holonomic Systems*, Invent. Math. **38** (1976), 33–53.

[4] H. Maass, *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Mathematics, vol. 216, Springer-Verlag, 1971.

[5] P.-D. Methée, *Sur les distributions invariantes dans le groupe des rotations de Lorentz*, Comment. Math. Helv. **28** (1954), 225–269.

[6] ———, *Transformée de Fourier de distributions invariantes* , C. R. Acad. Sci. Paris Sér. I Math. **240** (1955), 1179–1181.

[7] ———, *L'equation des ondes avec seconde membre invariante*, Comment. Math. Helv. **32** (1957), 153–164.

[8] M. Muro, *Microlocal analysis and calculations on some relatively invariant hyperfunctions related to zeta functions associated with the vector spaces of quadratic forms*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **22** (1986), no. 3, 395–463.

[9] ———, *Singular invariant tempered distributions on regular prehomogeneous vector spaces*, J. Funct. Anal. **76** (1988), no. 2, 317 - 345.

[10] ———, *Invariant hyperfunctions on regular prehomogeneous vector spaces of commutative parabolic type*, Tôhoku Math. J. (2) **42** (1990), no. 2, 163–193.

[11] ———, *Singular Invariant Hyperfunctions on the space of real symmetric matrices*, Tôhoku Math. J. (2) **51** (1999), 329–364.

[12] ———, *Singular Invariant Hyperfunctions on the space of Complex and Quaternion Hermitian matrices*, to appear in J. Math. Soc. Japan, 2001.

[13] T. Nomura, *Algebraically independent generators of invariant differential operators on a symmetric cone*, J. Reine Angew. Math. **400** (1989), 122–133.

[14] ———, *Algebraically independent generators of invariant differential operators on a bounded symmetric domain*, J. Math. Kyoto Univ. **31** (1991), 265–279.

[15] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81** (1949), 1–223.